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Symbol and Number

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Abstract

This paper explores the view that numbers are symbolically constituted, that numbers just are meaningful symbols. Such a view is what results if we take the conception of number spelled out by Husserl in the second part of his *Philosophy of Arithmetic* to be self-standing rather than supported by the conception of numbers as abstracted from sets. It will be argued that this latter conception is problematic in itself and, moreover, that it cannot be regarded as providing a foundation for the former.

Keywords: philosophy of arithmetic, Husserl, meaningful formalism, calculation, mathematical ontology

Each of the two parts of Husserl's *Philosophy of Arithmetic* (Husserl 1891) presents a separate conception of numbers. According to the first conception, numbers are abstractions from sets; according to the second conception, numbers are mirror images of symbols and cannot be thought of independently of a system of symbols. Husserl regarded the two conceptions as connected. In particular, he regarded the first conception as a foundation for the second: numbers as understood by the second conception form a superstructure built on top of numbers as understood by the first. Owing to our limitations in forming real, or authentic, representations of sets, we are for the most part relegated to this superstructure. In fact, arithmetic as we know it moves entirely within this superstructure.

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After giving a brief sketch of these two conceptions of number as spelled out in the *Philosophy of Arithmetic*, I shall question whether they in fact are connected in the way Husserl thought. A negative verdict leads me to explore the second conception of number as self-standing. Thus I shall explore the thesis that numbers are symbolically constituted objects, that numbers in effect just are meaningful symbols. The qualifier "meaningful" is important here: the philosophy of arithmetic to be explored is not formalism, namely the thesis that the real objects of mathematics are all finitary, such as strokes and rows of strokes; rather, the non-finitary notion of meaning will be invoked. Whether the resulting philosophy of mathematics has any affinity with Husserl's thought on mathematics, I shall not discuss in any detail; but it does seem to me to be an eminently phenomenological philosophy of mathematics. There are also interesting connections to the type theory of Per Martin-Löf, some of which will be noted.

1. According to the conception of number spelled out in the first part of the *Philosophy of Arithmetic* numbers are abstractions from sets (15-16.).[1] For instance, from the set {redness, the Moon, Napoleon} the number 3 is abstracted, and from the set of the Apostles the number 12 is abstracted. Husserl provides a detailed psychological description of set conception, or set constitution. What makes a representation into a representation of a set is a certain relation, or connection, that obtains between certain parts of the representation that are clearly distinguished from each other (what will become the elements of the set) (20). Much effort is spent on characterizing this relation, which Husserl calls the collective connection (*kollektive Verbindung*). It is, for instance, not the relation of compresence in one consciousness; nor the relation of temporal succession; nor that of sameness or of difference (64). Rather, it is a peculiar relation that is established by an intention directed towards clearly distinguished parts of the representation and that, as it were, holds these parts together. What are to become the elements of the set are all parts of the given representation; but, simply by being parts of a representation, they are still not represented as elements of a set; rather, it takes a second-order intention directed towards these elements to form the set representation (74).

Given such a set representation one can (according to Husserl's abstraction theory) disregard the particular nature of each of its elements and concentrate entirely on the collective connection itself (79). The result is a representation of what Husserl calls a plurality form (*Vielheitsform*), which we may symbolize in language as "something and something and something…" (80). Each occurrence here of the word "something" indicates an arbitrary object, and the ellipsis indicates indeterminacy, namely that the same pattern may continue arbitrarily long (81). Corresponding to the plurality form is the concept of plurality (Husserl is not clear about how to understand the relation between the form and the concept). The concept inherits the indeterminacy of the form (81). Any particular determination of the concept of plurality is a particular number, for instance 3 or 12. Borrowing a famous piece of terminology—though perhaps not the associated doctrine—from (Johnson 1921), we may call the concept of plurality a determinable whose determinations are particular numbers. Particular numbers are in turn described as species of the general concept of number, which concept we obtain by noting the similarity of particular numbers with each other (82). The relation between the general concept of number and particular numbers is therefore one of specification, whereas the relation between the concept of plurality and particular numbers is one of determination.

2. According to Husserl's other conception of number, numbers are the mirror images of numerical expressions. Thus, in the second part of the *Philosophy of Arithmetic* Husserl repeatedly speaks about a parallelism that obtains between number concepts and number signs (e.g. 228, 234, 237-241): the development of a system of number signs is at the same time a development of number concepts. The build-up of the signs mirrors the build-up of the concepts; in particular, the successive construction of number signs mirrors the succession of the number concepts. It is clear that the resulting conception of number is entirely different from the first conception. Here there is no reliance on sets, nor on abstraction. Rather, numbers are here conceived as elements of an ordered sequence, in which each element presupposes its predecessors. A number as abstracted from a set does, by contrast, not thus presuppose its predecessors, but only a set to abstract on. Whereas the first conception is thus a conception of cardinal number, the second conception is a conception of ordinal number.

The system of signs through which numbers are introduced provides a unique expression for each number (260). Numbers represented by these introductory expressions are called *normal* (261). For instance, in the standard decimal system the normal numbers are 0, 1, 2, 3,…, 10, 11, 12,…, 100, 101, 102,… Numbers not given in normal form are called *problematic*; examples are 2 + 3 and 7 × 5. Each problematic number presents us with a task, namely that of reduction to normal form (261). When the decimal numbers are taken to be the normal numbers, then the reduction of 2 + 3, for instance, yields 5, and the reduction of 7 × 5 yields 35. Reduction is thus just what we usually call calculation (258). A basic task of arithmetic is to delineate the various ways of forming problematic numbers and describe the methods of reducing numbers thus formed to normal numbers, that is, to describe methods of calculation (262). Given the parallelism between number signs and number concepts, calculation may be carried out on signs alone without regard to their content, since rules regarding signs will directly translate into rules regarding their content. The rules of manipulating signs are thus sound with respect to their intended content. Husserl considers in detail rules of calculation associated with the usual arithmetical operations—addition, multiplication, subtraction, division—and argues that these rules are indeed sound (264-272).[2]

3. A central thesis in the *Philosophy of Arithmetic* is that the two conceptions of number just described are in fact connected. The first conception describes numbers as they are given to us “authentically” (e.g. 15-16). In an authentic representation, a number is given to us just as it is, we see the “number in itself”. A finite mind is, however, limited in how large numbers it can represent authentically. Humans, in particular, may not be able to conceive authentically numbers greater than, say, ten or twelve. Since arithmetic presupposes arbitrarily large numbers, it therefore has to be based on some other conception of number, namely one according to which numbers are not authentically, but rather "symbolically" given (190-92). In a symbolic, or "inauthentic" representation, a number is given to us only indirectly, through a sign (193-94). The second conception of number is just such a symbolic conception, and it, according to Husserl, is the conception of number assumed in arithmetic.

Although arithmetic thus assumes numbers to be symbolically, and not authentically, represented, it would seem odd to say that symbolic, or inauthentic, representations lie at the foundations of arithmetic; for this would be to say that the science of number is based on representations in which numbers are in fact not properly given to us. Rather, one would expect that authentic representations of numbers in some way provide a foundation for symbolic representations of numbers and, thereby, also for arithmetic.

It is, however, difficult to see precisely how Husserl's account of symbolically given numbers is to be grounded in his account of authentically given numbers. The impossibility for us to form authentic representations of numbers larger than, say, twelve stems from the impossibility for us to form authentic representations of sets with more than that number of elements. For larger sets we are confined to symbolic representations. For instance, the set representation that we form when we enter a lecture hall and see a large audience is, in the typical case, only symbolic. Husserl provides a detailed account of such symbolic set representations (195-218). This account does, however, not play any role in Husserl's account of symbolically given numbers (contrary to what Husserl seems to suggest at 222-23). In particular, a symbolic representation of a number is not, according to Husserl's account, obtained by abstraction from a symbolically conceived set. Rather, Husserl's introduction of symbolic numbers follows an entirely different plan, with no reliance on symbolic sets or abstraction.

It is true that, at the initial stages of the introduction of symbolic numbers authentic number representations are involved. Symbolic numbers are introduced through a (potentially infinite) system of signs. Husserl prefers a system in base X that employs addition, multiplication, and exponentiation for forming numbers larger than X (228-33). Examples of numbers in this system are therefore X, $(2 \times X) + 3$ and $(3 \times X^2) + 4$ (assuming that X is greater than 4). When X is 10, these symbolic numbers are just decimal numbers written out completely. Authentic number representations are involved in the description of these symbolic numbers through the requirement that the base number X be authentically given. That is, there is some freedom in choosing X, but it has to be a number of which we can form an authentic representation.

Once in place, this decimal-like system yields, according to Husserl, a unique representative for each number in itself (*Zahl an sich*). That is, we get a unique representative for each number as it is given in an authentic representation, or rather: as it would be given in an authentic representation, if we could form such a representation (260). But it is unclear how the existence of such a one-to-one correspondence can be justified for numbers that we cannot authentically represent to ourselves. The thought is perhaps that since there is a one-to-one correspondence for each of the numbers up to X, there is one as well for any number built up from these numbers, just as the numbers in the described system are. But this is a *non sequitur*. If, for instance, we can have no authentic representation of numbers greater than twelve, then how can we possibly know that, say, $(3 \times 10^2) + (7 \times 10) + 1$ is the unique representative of a number in itself, the true three-hundred-and-seventy-one? Although the symbolic number in question is built up from numbers—3, 10, 2, 7, 10, 1—each of which may be taken to correspond uniquely to a number in itself, we have no guarantee that the same can be said of a number formed through addition, multiplication, and exponentiation from these. These operations may well take us out of the range of authentically representable numbers; and for any number outside that range, we have no way of ascertaining whether it has a unique representative in the system of symbolic numbers.

Since sufficiently large numbers are inaccessible to us, we thus seem to have no way of assessing claims regarding their relation to symbolic numbers. A similar problem affects Husserl's whole conception of numbers as abstractions from sets. If we are in principle denied knowledge of numbers sufficiently large, then what right do have to claim, for instance, that there are infinitely many numbers? For all we know, the sequence of numbers in themselves might terminate at some point outside the range of authentically representable numbers. And how can we be certain that the numbers are linearly ordered? Perhaps at some point far out in the number sequence there is a number that has two immediate successors. If only an initial segment of the number sequence is accessible to us, then we have no right to exclude the possibility of such a situation. That no such situation arises in the part of the number sequence that we do have access to is no proof that it cannot arise in some part that we do not, and indeed cannot, have access to.

There is anyway something strange in developing an account of number and operations on numbers and then go on to say, as Husserl does (190-92), that arithmetic is in fact grounded on an entirely different account of number. Indeed, Husserl thinks that if we could conceive of any number in the way the first account describes, then there would be no arithmetic, since then all relations among numbers would be immediately evident to us (191).[3] The existence of arithmetic thus shows that the first account cannot be our only account of number. The difficulty in seeing how the first account might serve as a foundation for the second account suggests, to my mind, that we might as well forget about it altogether as an account of number. What is presented in the first part of *Philosophy of Arithmetic* should be regarded as an account of finite sets, not as an account of number. We are there given a detailed account of the conception of finite sets; but the abstraction theory of number that is based on it is idle, since it cannot serve as a foundation of arithmetic.

I should emphasize that my criticism here does not concern the distinction between authentic and symbolic representations as such. This is clearly an important distinction, as witnessed by its refined reappearance in Husserl's epistemology from the *Logical Investigations* onwards as the distinction between an intention and its fulfilment. The problem in the *Philosophy of Arithmetic* is not the invocation of this distinction as such, but rather the use Husserl makes of it: he posits objects of which authentic representations are in principle excluded. He says that there are such and such objects, although we shall never be able to see them with evidence, we shall never have proper knowledge of them. The positing of such knowledge-transcendent objects is, to my mind, quite foreign to phenomenology. The phenomenological point of view is a first-person point of view, so a phenomenologist's positing of objects should always be accompanied by a description of how such objects can be given to us. Husserl does quite the opposite when he posits certain objects and, at the same time, denies that they can ever be given to us.[4]

4. We are thus led to explore the prospects of Husserl's second account of number as self-standing, without the spurious support from the account of numbers as abstractions from sets. The main tenet of the resulting philosophy of arithmetic—however it is worked out in detail—is that numbers are symbolically constituted objects. Numbers are given by a system of meaningful symbols; not, however, in the sense that the "numbers in themselves" are mirror images of these symbols—as Husserl seems to have held—but in the sense that the numbers are these very symbols. There is no number in itself apart from the meaningful symbol that you see on the page in front of you. Apart from a short remark at the beginning of section 6 below, I shall not discuss here whether this can be taken to be Husserl's own philosophy of arithmetic at any point of its development. It does, in any event, seem to me to be an eminently phenomenological philosophy of arithmetic. Working out the details will require considerable effort. Here I wish only to note some possible sources of inspiration from the second part of the *Philosophy of Arithmetic*.

Symbolically constituted numbers are, as already noted, introduced through a system of signs. Husserl considers several alternative sign systems: apart from the base-X system, which he prefers, also a unary system (228); a "non-systematic" system (*sic*), where numbers are given only provisional definitions in terms of auxiliary number signs and addition (224-5); and a system where the numbers are generated in the natural order, but where each number is named quite independently of the names of smaller numbers, unlike what is the case in the base-X and the unary system (226-7).

Since we regard symbolically constituted numbers as self-standing, we cannot follow Husserl in his preference for the base-X system. This system, namely, relies on the arithmetical operations of addition, multiplication, and exponentiation; and we cannot take an understanding of these for granted when we first introduce the numbers. Husserl could perhaps do so, since he took symbolic numbers to correspond to numbers as abstractions from sets, and for these he had given an account of the basic arithmetical operations (182-90). We cannot do the same, however, since we have rejected this account of number and wish to consider symbolic numbers as self-standing. (The positional system in base X, in which, for instance, $(3 \times X^2) + (7 \times X) + 1$ is written "371" does not avoid the reliance on addition, multiplication and exponentiation either, since it is merely an abbreviation of the more long-winded base-X system that Husserl employs.[5])

From the foundational point of view, a unary system must be preferred. It should, however, not be formulated as Husserl formulates it (228), in terms of successive additions of 1. The numbers would then be introduced as 1, 1+1, 1+1+1, …, so we should again be relying on addition. It is true that we here invoke only a special case of addition, namely where the second argument is 1. But addition as such is a binary function defined on all pairs of numbers, so we need to see (1+1)+1, say, as an instance of the general form $m + n$. Since it is not by this general form that the numbers are generated, we cannot regard addition as being defined simultaneously with the introduction of the numbers. Rather, the definition of addition has to wait until we have explained how the numbers are generated. For the generation of the numbers in the first place, we should rely on a successor function and the basic number 0. (These primitive notions may be regarded as being explained simultaneously with the introduction of the numbers.) The numbers are thus generated as 0, **s**(0), **s**(**s**(0)), **s**(**s**(**s**(0)))… Addition, multiplication, and exponentiation can then later be defined by well-known recursion equations.[6]

5. We thus take numbers to be introduced as 0, **s**(0), **s**(**s**(0)), **s**(**s**(**s**(0)))… It is, however, obvious that not all numbers are given in this unary form. The number 371 in the decimal positional system, for instance, is not so given, nor is, say, 7×5. Clearly, we do not want to be forced to say that these are not numbers. They do not look like numbers as introduced by the unary system, but they are numbers nevertheless. An important idea in the second part of the *Philosophy of Arithmetic* is the distinction between what Husserl calls normal and problematic numbers (261). Normal numbers are numbers in introductory form, whereas problematic numbers are numbers in non-introductory form. Employing this terminology we can say that 371 and 7×5 are indeed numbers, but problematic numbers. They are not called problematic because their status as numbers is somehow problematic. Rather, they are called problematic because each poses—or, better, *is*—a problem, or a task (*Aufgabe*), namely that of reduction to normal form. In particular, each of the numbers 371 and 7×5 is a task of reduction to a number in the form of 0 followed by some number of iterations of the successor function.

Husserl's immediate aim in introducing the distinction between normal and problematic numbers is to be able to say when two symbolically given numbers are identical. It is natural to stipulate that not only any problematic number, but also any normal number is a task: a normal number is the trivial task that is solved by itself. In terms of reduction to normal form, we thus stipulate that a normal number reduces to itself. As a consequence of this stipulation, it will make sense to speak, for any number, of its reduction to normal form. Two symbolically constituted numbers can then be said to be identical if they reduce to the same normal number. For instance, the numbers 7×5, $27 + 8$, 35 and $\mathbf{s}^{35}(0)$ are all identical to each other, since each reduces to $\mathbf{s}^{35}(0)$ that is, 0 followed by thirty-five iterations of the successor function.

Contained in these stipulations are both a criterion of application and a criterion of identity for numbers. Recall that a criterion of application associated with a concept *C* determines what it is for an object to fall under *C*; and that a criterion of identity associated with *C* determines what it is for objects falling under *C* to be identical.[7] We take numbers to be introduced as 0, **s**(0), **s**(**s**(0)), **s**(**s**(**s**(0)))… These are the normal numbers. By the introduction of numbers we thus know what a normal number is. A number quite generally can then be said to be a task of reduction to a normal number. This is the criterion of application for numbers. The criterion of identity says that numbers are identical if they reduce to one and the same normal number.

These criteria of application and identity for numbers agree with those given by Per Martin-Löf as part of the so-called meaning explanations for his constructive type theory (Martin-Löf 1984). The numbers—or, more precisely, the natural numbers—are there an instance of the more general concept of a type of individuals. A type *A* of individuals is defined by laying down how the elements of normal form of that type are constructed. Thus, one defines a type of individuals by displaying a mode of generation of its normal-form elements, in a manner similar to how numbers were introduced above as 0, **s**(0), **s**(**s**(0)), **s**(**s**(**s**(0)))… In this context elements of normal form are usually called "canonical elements". From the definition of *A* we thus know what the canonical elements of *A* are. The criterion of application for *A* is then formulated as follows:

> an element *a* of a set *A* is a method (or program) which, when executed, yields a canonical element of *A* as result (Martin-Löf 1984, 9)

A canonical element of a set is a method that yields itself as result when executed, hence any canonical element of *A* is also an element of *A* according to this criterion. The criterion of identity for *A* is formulated as follows:

> two arbitrary elements *a*, *b* of a set *A* are equal if, when executed, *a* and *b* yield equal canonical elements of *A* as results (ibid.).

The similarity between these explanations and those we extracted above from the *Philosophy of Arithmetic* should be obvious. We took a number to be a problem, or task, of reduction to normal form, and we may think of such a problem as a programme which, when executed, yields a normal-form number; and we took numbers to be identical if they reduce to the same normal-form number, that is, the same canonical number. This similarity may not be a coincidence: the thesis that mathematical objects quite generally (not only numbers) are symbolically constituted is a philosophy of mathematics that is quite congenial to the spirit of constructive type theory.

The task presented by a number is solved by calculation. Thus the reduction of 7×5 to $\mathbf{s}^{35}(0)$ is just the calculation of 7×5, a calculation whose result is $\mathbf{s}^{35}(0)$. Calculation itself may be regarded as the unravelling of definitions. For instance, 7 is defined as $\mathbf{s}(6)$, 6 is defined as $\mathbf{s}(5)$, 5 is defined as $\mathbf{s}(4)$, etc. Moreover, we have definitions of functions such as addition, multiplication, and exponentiation. Employing these definitions we reduce a number by continued substitution of *definiens* for *definiendum*.[8] Thus, from the defining equations of multiplication we find that $7 \times \mathbf{s}(4)$ reduces to $(7 \times 4) +7$, and from the defining equations of addition that $(7 \times 4) + \mathbf{s}(6)$ reduces to $\mathbf{s}((7 \times 4) + 6$. Likewise we find that $(7 \times 4) + 6$ reduces to $\mathbf{s}((7 \times 4) + 5)$. Continuing this procedure we shall eventually reach $\mathbf{s}^{35}(0)$. The conception of calculation as the unravelling of definitions, later made precise by Kleene, Curry, Martin-Löf and others,[9] is not quite what one finds in the *Philosophy of Arithmetic*; but it can be found in *Logical Investigations* VI §18 (Husserl 1901), where Husserl describes in some detail the reduction of the number $(5^3)^4$ to unary form. (At this point, therefore, Husserl seems to prefer the unary form as the normal form; indeed, in the cited section he in effect says that a decimal number is a task of reduction to unary form.)

6. The section of the *Logical Investigations* just cited is part of a discussion of the notions of intention and fulfilment. Husserl suggests that we may regard the substitution of *definiens* for *definiendum* as a step of partial fulfilment;

complete fulfilment is then reached with what Curry calls the ultimate *definiens*, an expression that can no longer be reduced. Given the understanding of such a process of substitution as a calculation, this section of the *Investigations* thus suggests that we think of the relation between a problematic and a normal number as the relation between an intention and its complete fulfilment. In the theory of fulfilment in the *Investigations* complete fulfilment is achieved by the presence of the intended object itself. And indeed, in the cited section Husserl speaks of the end result of the process of substitution as the "number itself". Here, therefore, the number itself is just the meaningful expression that is the number in its unary form, quite in line with the doctrine currently being explored. Normal numbers have thus taken over the role of numbers in themselves. There is, for instance, nothing beyond the meaningful symbol **s**(**s**(**s**(0))) that is the number three itself. Rock bottom has been reached already with this normal form.

It was already noted that the pair of notions of intention and fulfilment has a precursor in the pair of notions of inauthentic, or symbolic, and authentic representations. That we may think of the relation between a problematic number and the normal number to which it reduces in terms of the former pair of notions suggests that we may also think of it in terms of the latter pair of notions. For Husserl, of course, a symbolically constituted number cannot be authentically represented: for him, only numbers as abstractions from (authentically represented) sets can be so represented. But, for us, since we here take normal-form numbers to play the role of numbers in themselves, it is natural to say that such numbers are authentically given, whereas numbers in problematic form are inauthentically given. That is, it is natural to say, for instance, that the number **s**(**s**(**s**(0))) is here authentically given, whereas 2 + 1 is the same number inauthentically given.

We then recognize a phenomenon that was central in Husserl's original employment of this terminology: of sufficiently large numbers it is physically impossible for us to achieve an authentic representation, though we shall always be able to construct an inauthentic representation. Take, for instance, the number $10^{10^{10}}$. We have no choice but to present this number inauthentically, since an authentic representation of it, consisting of a sequence of **s**'s appended to 0, is out of reach for us.[10] (This point does not depend on the use of the unary system: when numbers get sufficiently large, we cannot write them out in the base-X positional system either.) That an authentic representation of $10^{10^{10}}$ is out of reach for us is, however, not to say that we need to withhold judgement as to whether this number in fact exists. The number $10^{10^{10}}$ is a well-defined problem, we know precisely what it means to calculate it. And given our criterion of application for numbers, this is all that is required for us to have the right to say that this is indeed a number, that this number exists. The contention that sufficiently large numbers can only be inauthentically represented therefore does not, for us, lead to any doubts regarding the existence of large numbers, as it does in Husserl's original theory.

7. A final pair of notions that we shall invoke in elucidating the relation between problematic and normal numbers is the pair of sense and reference. It has been noticed by several readers of (Frege 1892) that the relation between the sense of an expression and its reference may, at least in some cases, be understood as the relation between a programme, or task, and the result of its execution.[11] Frege himself suggests this idea in a *Nachlass* piece:

> "4", "2^2", "$(-2)^2$" are only different signs for the same, whose difference merely indicates the different ways along which one may reach the same thing. (Frege 1983, 95)

Thus, different signs for the same thing indicate different ways leading to that thing. But a way leading to a certain goal is just what a programme or method is.[12] In light of the sense/reference-distinction, the quoted passage can be read as saying that "4", "2^2", "$(-2)^2$" express different senses, whose difference indicates different ways of reaching the same reference. Indeed, the root of the words "*Sinn*" and "sense" has to do with locomotion and can mean way or route.[13] A remnant of this root meaning can be seen when these words are used today to mean direction. Taking advantage of this etymology, we might say that the sense of an expression charts out a route to its reference. In a similar way, a problematic number, being a task, or a programme, can be said to chart out a route to a normal number, namely the route that we follow when solving the task, or executing the programme. It is therefore natural to say that a problematic number stands to the normal number to which it reduces as sense stands to reference. The fact that syntactically different problematic numbers may evaluate to the same normal-form number then reflects the fact that different senses may determine the same reference.

The reduction of a problematic number to its normal form can be represented by a sequence of numbers in which each element is obtained from the previous one by reduction on the basis of definitions. For instance, the reduction of 2 + 1 to $\mathbf{s}^3(0)$ may be represented as the sequence

$$(2 + 1) \to (2 + \mathbf{s}(0)) \to \mathbf{s}(2 + 0) \to \mathbf{s}(2) \to \mathbf{s}(\mathbf{s}(1)) \to \mathbf{s}(\mathbf{s}(\mathbf{s}(0)))$$

Although all elements of this sequence are identical, they are also all of them syntactically different from each other. In the terminology of sense and reference we may say that we here have different senses determining one and the same reference. Frege would certainly say that this reference is an object—the number three—that exists apart from this sequence of senses. For us this is not so. For us, the object in question is the final element of the sequence, viz. $\mathbf{s}(\mathbf{s}(\mathbf{s}(0)))$. The reference in this case thus resides at the same level as the senses determining it. Our adoption of the sense/reference-terminology was suggested by the understanding of the sense of an expression as a programme the result of executing which is the corresponding reference. According to our criterion of application for numbers, any number is a programme. A normal-form number, in particular, is a programme the result of executing which is itself. It must therefore be regarded as a sense that determines itself as reference. In a normal-form number we thus have a collapse of sense and reference.

8. It should be clear by now that the philosophy of arithmetic that has been explored here is not the doctrine that the objects of arithmetic are meaningless signs, a view Husserl himself criticizes in the *Philosophy of Arithmetic* (e.g. 170-178). The doctrine explored is rather that the objects of arithmetic are meaningful symbols. Unlike objects such as trees, books, and people, numbers are symbolically constituted: a number is nothing apart from the meaningful expression that presents it; the number is there in the expression. We can of course consider an expression as a formal entity devoid of sense, but in that case, what we see is not a number, but a certain "formal object" (to use another piece of terminology from Curry). In the usual object-directed attitude, by contrast, the expression is regarded as meaningful, and then we see, for instance, the number 7 × 5. What the meaning of this expression is we in effect specified when giving the criterion of application for numbers: the meaning is specified by saying that 7 × 5 is a programme for obtaining a number in normal form. That this is indeed a notion of meaning was brought out by our invoking Frege's sense/reference-distinction to illuminate the relation between a programme and the normal number that it yields upon calculation.

A natural question now is whether the view explored here can be extended to all of mathematics: can we say that not only the objects of arithmetic, but in fact all mathematical objects are symbolically constituted? What we cannot say, it seems, is that all mathematical objects are senses as explicated here, viz. programmes of reduction to normal form; for it should not be expected that all mathematical objects can be regarded as such programmes. In particular, a doubt may arise concerning functions. A programme as understood here is something that can be calculated, or reduced, to normal form. We do, however, not calculate a function such as the multiplication function, ×, in isolation. What is calculated is rather 7 × 5 or 3 × 10 or any other result of supplying the multiplication function with two numbers as arguments.

Although a function is thus not a programme in the specified sense, that is, an object of calculation, and therefore not a sense as explicated here, we may nevertheless regard it as a meaningful symbol. For instance, we can take the meaning of × to consist in the fact that when supplied with two numbers as arguments we get a programme that can be calculated to normal form, say by relying on the definition of expressions of the form $m \times n$. Assuming that we have explained what it is to be a function quite generally in some such way,[14] then we should have made important steps towards the justification of viewing functions as symbolically constituted objects. That in turn would be an important step towards extending the work Husserl began in the second part of the *Philosophy of Arithmetic* to a general philosophy of mathematics.

NOTES

[1] All page references given without any further specification are to the *Husserliana* edition of the *Philosophy of Arithmetic* (Husserl 1891).
[2] For a more detailed overview of the contents of the *Philosophy of Arithmetic*, see (Centrone 2010, ch. 1).
[3] This claim is stronger than the claim that there is no arithmetic for an infinite intellect; for it is here presupposed only that *any* number is authentically conceivable, not that they are all authentically conceivable at once.
[4] A conclusion similar to that reached here is also reached in (Miller 1982, 77): "Large numbers would seem to have no being whatsoever, if the act which is their unity cannot be performed."
[5] Husserl's preference for the more long-winded system stems, probably, not only from the fact that it is more transparent, but also from the fact that the positional system presupposes 1 and 0: from Husserl's point of view these are not unproblematically called numbers (129-34).
[6] The need for using a separate successor function rather than the special case of addition, $m + 1$, was seen by Dedekind. In (Dedekind 1888), where the generation of the numbers by means of the successor function is made precise (albeit by the use of impredicative methods), one also finds recursive definitions of addition, multiplication, and exponentiation. Husserl had read this work, was impressed by its rigour, but found that "in its strange artificiality, it strays far from the truth" (125).
[7] A concept with which both a criterion of application and a criterion of identity are associated is usually called a *sortal* concept. The notion of a criterion of identity is often traced back to (Frege 1884, §62). The term "criterion of application" stems from (Dummett 1973, 74).
[8] If the definition of a function is given in terms of variables, then besides substitution we also a need to rely on instantiation. For instance, from the definitional equation $x + \mathbf{s}(y) = \mathbf{s}(x + y)$ we get $7 + \mathbf{s}(4) = \mathbf{s}(7 + 4)$ by instantiation.
[9] See (Kleene 1952, esp. §54) and (Curry and Feys 1958, esp. ch. 2E).
[10] Consider the problem of printing this many **s**'s. A quick calculation shows that if we print, say, 1000 **s**'s per second, then a conservative estimate of the number of years required is the number written in decimal notation as 1

followed by a milliard zeros. In comparison, the number of years written in decimal notation as 1 followed by 11 zeros is already larger than the estimated age of the universe.

[11] See, for instance, (Dummett 1991, 123), stemming from lectures given in 1976, or (Tichý 1988). An unpublished lecture by Martin-Löf given in 2001 should also be mentioned, since it has been important for the current presentation.

[12] The Greek word *methodos* is a combination of *meta* and *hodos*, and the primary meaning of this latter word is way or route.

[13] See Pokorny's *Indogermanisches etymologisches Wörterbuch* s.v. "sent-"

[14] A part of the general explanation would be: *f* is a function if *f*(*n*) is a number whenever *n* is a number. This explanation presupposes that all functions are total, namely that *f*(*n*) is always a programme that in fact yields a normal number upon evaluation, whichever number *n* might be. It would therefore not work for Husserl's conception of functions (or operations, in his terminology) if it is right, as Centrone (2010) has stressed, that Husserl allows partial functions.

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